Evaluation of Solution Procedures for Material and/or Geometrically Nonlinear Structural Analysis

JAMES A. STRICKLIN* AND WALTER E. HAISLER†
Texas A & M University, College Station, Texas
AND
WALTER A. VON RIESEMANN‡
Sandia Corporation, Albuquerque, N. Mex.

This paper presents an assessment of the solution procedures available for the analysis of inelastic and/or large deflection structural behavior. A literature survey is given which summarizes the contribution of other researchers in the analysis of structural problems exhibiting material nonlinearities and combined geometric-material nonlinearities. Attention is focused at evaluating the available computational and solution techniques. Each of the solution techniques is developed from a common equation of equilibrium in terms of pseudo forces. The solution procedures are applied to circular plates and shells of revolution in an attempt to compare and evaluate each with respect to computational accuracy, economy, and efficiency. Based on the numerical studies, observations and comments are made with regard to the accuracy and economy of each solution technique.

Nomenclature

$\lfloor \; \rfloor$ = row matrix
$\{ \; \}$ = column matrix
$[\;]$ = square matrix
C = scalar coefficient in Eq. (12)
K = matrix of stiffness coefficients
P = load parameter ($P_{max} = 100$)
$\bar{P}$ = normalized load vector
P' = generalized forces due to applied loads [Eq. (4)]
Q = pseudo force due to nonlinearities
Z = scalar coefficient in Eqs. (11) and (12)
f = force unbalance in equation of equilibrium [Eq. (18)]
q = generalized displacements
Δ = increment

Superscripts
$(\dot{\;})$ = differentiation with respect to load parameter P
NL = contribution due to geometric nonlinearities
P = contribution due to material plasticity

Introduction

THE finite element method has been applied quite successfully to a wide range of linear structural problems and, recently, the method has been extended to include geometric and material nonlinearities. While the analysis of linear problems is relatively straightforward, the nonlinear problem is considerably more difficult. Although a great number of papers have been published on the analysis of nonlinear behavior, a thorough survey of this available literature indicates that there is considerable uncertainty regarding such important questions as which yield criteria are the best, which flow rule is correct, how should unloading be treated, and so on. Perhaps the most perplexing and disturbing point has been that most of the computational procedures currently available are very inefficient and require excessive amounts of computer time when applied to practical, large scale structural systems. The long computer run times (ranging from 30 min to several hours) severely limit the use of these nonlinear codes as design tools for the majority of users.

The present study concerns itself with evaluating the solution techniques for the nonlinear problem with regard to accuracy and computational economy. The equations of equilibrium are first developed in Lagrangian coordinates with the nonlinearities being treated as pseudoforces. Various solution forms which follow directly from the equilibrium equations are described in terms of pseudoforces and/or tangent stiffness matrices. These include incremental forms, Newton-Raphson, self-correcting forms, and others. The equations are developed in general form with applications made to a number of plate and shell of revolution problems. Based on the numerical studies, some general comments and recommendations are made with regard to the accuracy and economy of each method tested.

This paper reports only a small portion of the research underway by the authors in the area of combined large deflection elastic-plastic behavior. Another paper[1] considers the development of efficient computational algorithms for the evaluation of the pseudoforce and tangent stiffness terms. Many of the details omitted here for lack of space are reported in Ref. 2.

The next section presents a literature survey of solution techniques available for the nonlinear problem. A brief summary of the development of the equilibrium equations, the procedure for evaluating the pseudoforces and tangent stiffness matrix contributions, and the plasticity relations is then presented. Each of the solution techniques and its application to a number of problems is presented in detail. A final section presents recommendations and observations relative to the accuracy and efficiency of each procedure.

Literature Survey

Survey papers pertaining to geometric nonlinearities have been compiled by several researchers[3,4] including some rather complete literature surveys by the authors.[5-7] To avoid duplication, the present survey of the literature will only consider the solution of plasticity problems and combined geometric-material nonlinearities.

Received April 7, 1972; presented as Paper 72-353 at the AIAA/ASME/SAE 13th Structures, Structural Dynamics, and Materials Conference, San Antonio, Texas, April 10–12, 1972; revision received October 24, 1972. Research supported under Sandia Contract 82-5617, NASA Grant NGL-44-001-044, ONR Contract N00014-68-A-0308-0004, and U.S. Atomic Energy Commission.

Index category: Structural Static Analysis.

* Professor, Department of Aerospace Engineering. Member AIAA.
† Asssistant Professor, Department of Aerospace Engineering. Associate Member AIAA.
‡ Staff Member. Member AIAA.

The survey of the literature will be presented according to the method used in the solution of the resulting equations of equilibrium. For the present discussion, the only types of nonlinearities considered are those due to plastic strains; geometric nonlinearities are considered later. The basic equations of equilibrium are of the form

$$[K]\{q\} = \{P'\} + \{Q(q)\} \tag{1}$$

where $[K]$ = global stiffness matrix, $\{q\}$ = column matrix of generalized displacements, $\{P'\}$ = generalized forces due to applied loads, $\{Q(q)\}$ = column matrix of pseudo forces due to plastic (initial) strains.

Ideally, Eq. (1) is exactly satisfied. However, with many methods, the equations of equilibrium are only satisfied to within a certain degree of accuracy. For this reason it is convenient to define an unbalance of force term given by

$$\{f\} = -[K]\{q\} + \{P'\} + \{Q\} \tag{2}$$

where $\{f\}$ is the amount of out of balance in force which exists.

The first class of solution procedures is that which satisfies or at least attempts to satisfy the equations of equilibrium exactly. For this class, $\{f\} = \{0\}$.

The first procedure used to satisfy $\{f\} = \{0\}$ is the method of successive approximations. In this method the load is increased in increments and, at each increment of load, iteration until convergence is performed using the recursion relation

$$[K]\{q_i\} = \{P_i'\} + \{Q_{i-1}\} \tag{3}$$

where i is the ith iteration and $\{Q_{i-1}\}$ are the pseudo plastic forces based on the generalized displacements at the $(i-1)$th iteration. The process is usually started by a simple linear extrapolation based on previous solutions. Mendelson and Manson[8] first proposed the method of successive approximations in 1959. Others to discuss the method are Argyris,[9] De Donato,[10] Havner,[11] Marcal,[12] and Witmer and Kotanchik.[13] The main difficulties which are encountered with the method are the very slow rate of convergence for large plastic strains and the tendency to converge to the incorrect answer for elastic-perfectly plastic materials. De Donato[10] shows that his iterational procedure fails for perfectly plastic materials. Havner[11] stipulates a monotonically increasing stress strain curve and shows that the iterational procedure will converge but does not show the rate of convergence. Marcal[12] shows that the method of successive approximations converges to the incorrect answer for elastic-perfectly plastic materials. Witmer and Kotanchik[13] use this method to solve complex shell of revolution problems and state in their conclusions that some technique is needed to accelerate convergence. Zienkiewicz, Valliappan, and King[14] have presented a method of successive approximations which they call the initial stress approach. In Ref. 15, it is shown that the method of initial stress is in essence the same as the method of successive approximation.

Another solution procedure of the class $\{f\} = \{0\}$ is the Newton-Raphson procedure first used by Oden and Kubitza.[16] They solved for the deflection of a square plate with external pressure. The only type of stress-strain behavior considered was elastic-plastic with appreciable strain hardening but without unloading. It will be shown in the present research that consideration of elastic unloading causes the Newton-Raphson procedure to fail to converge in many cases.

Another solution procedure which satisfies the equation of equilibrium exactly is the minimization of the total potential energy as presented by Stanton and Schmit.[17] In Ref. 17 the authors use deformation theory instead of the more accurate and commonly used incremental theory. This particular solution procedure is not evaluated in the present study.

A second class of solution procedures and apparently the most popular to date contains those methods which solve the equation $\{\dot{f}\} = \{0\}$ where the dot indicates differentiation with respect to a load parameter. For example, the load vector $\{P'\}$ may be written as

$$\{P'\} = P\{\bar{P}\} \tag{4}$$

where P is some convenient normalizing term which, for convenience, is taken as ranging from 0 to 100. Differentiation is then performed with respect to P. From Eq. (2) the expression for $\{\dot{f}\} = \{0\}$ becomes

$$[K]\{\dot{q}\} = \{\bar{P}\} + \{\dot{Q}\} \tag{5}$$

A frequently used solution procedure for Eq. (5) is to use an Euler forward difference for $\{\dot{q}\}$ and a backwards difference for $\{\dot{Q}\}$. Thus, for any increment

$$[K]\{\Delta q\}_i = \Delta P\{\bar{P}\} + \{\Delta Q\}_{i-1} \tag{6}$$

It is noted that Eq. (6) has a truncation error of ΔP.

Another form which is equivalent to Eq. (6) is in terms of the total displacement and total pseudoplastic forces

$$[K]\{q_i\} = \{P_i'\} + \{Q_{i-1}\} \tag{7}$$

The form given by Eq. (7) is obtained by expanding $\{\dot{Q}_i\}$ in a Taylor's series expansion and retaining only the constant term

$$\{\dot{Q}_i\} = \{Q_{i-1}\} + \Delta P\{\dot{Q}_{i-1}\} \tag{8}$$

It is seen that Eq. (7) has an error term of ΔP. Equations (6) and (7) are essentially the same except that one is written in terms of increments of displacements and one is written in terms of total displacements. This is easily seen by noting that Eq. (6) may be obtained by subtracting Eq. (7) at increment $i-1$ from Eq. (7) at increment i.

Finally, returning to Eq. (5), chain rule differentiation may be used on the pseudoforce term to obtain

$$\{\dot{Q}\} = [\partial Q_i/\partial q_j]\{\dot{q}\} \tag{9}$$

Substituting Eq. (9) into Eq. (5) results in

$$[[K] - [\partial Q_i/\partial q_j]]\{\dot{q}\} = \{\bar{P}\} \tag{10}$$

Equation (10) may be solved by any of a large number of numerical procedures. However, the most common procedure is a simple forward integration procedure.

Gallagher, Padlog and Bijlaard[18] used Eq. (7) to solve plasticity problems as early as 1962. In Ref. 18 they presented two approaches for computing the plastic strains. These methods are referred to as the constant stress and constant strain approaches. It was shown that the constant stress method encounters numerical instabilities. These numerical instabilities cannot be overcome by reducing the load increment. In fact, the reverse occurs in that the smaller the load increment, the sooner the numerical instabilities occur. This approach or its equivalent given by Eq. (6) has also been pursued in Refs. 9, 19, and 20. The general consensus of opinion gained from these references is that the initial strain approach given by Eqs. (6) or (7) is very slow to converge with reduction in the size of the load increment.

Isakson, Armen, and Pifko[21] present a slight modification of the solution procedure given by Eq. (7). In their method, called the predictor method, the pseudoplastic force terms are estimated based on their values at previous loads. For example, this may be accomplished by a simple linear extrapolation. The use of an extrapolation procedure does not seem to introduce numerical instabilities into the solution procedure. For geometric nonlinearities, on the other hand, solutions by Eq. (7) become unstable for moderate nonlinearities and any type of extrapolation procedure tends to hasten the instability.[5]

The incremental procedure as given by a simple forward difference solution of Eq. (10) was developed by Pope,[22] Swedlow and Yang,[23] Marcal and King[24] and Yamada.[25] The incremental procedure was further developed by Armen et al.[19] where Ziegler's modification of Prager's hardening rule was used and problems involving cyclic loading were solved. In Ref. 19 the shear lag problem was solved by both the initial strain and the incremental method. Results show that the incremental procedure converges quite rapidly compared with the initial strain procedure. Further comparisons of the two methods are given in Refs. 26 and 27 with the incremental tangent stiffness method being judged the more useful.

Felippa[28] solved Eq. (10) by a two step method where the tangent stiffness matrix is evaluated one-half increment forward using a simple forward difference procedure. This is sometimes referred to as a chord stiffness matrix. Reference 28 also implies that the problem may be formulated in terms of a first order nonlinear differential equation.

Richard and Blacklock[29] solved Eq. (10) by the Runge-Kutta procedure and presented an inverse Ramberg-Osgood curve for the uniaxial stress-strain relation. They show in Ref. 29 that excellent results may be obtained using a small number of increments with the Runge-Kutta solution procedure. However, it should be pointed out that the equations must be solved four times for each increment of load and hence, this procedure may be quite time consuming.

The advantage of writing the equilibrium equations in the form given by Eq. (10) is that this form opens the door to a very large class of solution techniques. These include simple forward differences, predictor-corrector schemes, Runge-Kutta methods, and so on. However, it has been shown by Haisler[5] that for the geometrically nonlinear case, some of these techniques are not applicable because of accuracy or stability considerations. Consequently, this research will only make use of the Euler forward integration method to solve Eq. (10).

Two other classes of solution procedures which have only recently been formulated[6,30,31] are evaluated in the present study. These solve the equations of the form

$$\{\dot{f}\}+Z\{f\}=0 \tag{11}$$

and

$$\{\ddot{f}\}+C\{\dot{f}\}+Z\{f\}=0 \tag{12}$$

where C and Z are scalar quantities. These forms are referred to as self-correcting forms[31] as the unbalance in force returns to zero whereas $\{\dot{f}\}=0$ tends to drift away from the true solution For $Z=1/\Delta P$ in Eq. (11) the procedure reduces to the form given in Refs. 32 and 33 which may be interpreted as an incremental form with a one step Newton-Raphson correction.[33] Solutions of the class given by Eq. (12) have not been used for the solution of plasticity problems but have been found to be economical for geometrically nonlinear problems.[6,31]

The literature on combined material and geometric nonlinearities is quite limited and is represented by Refs. 28, 32, 34–37. Further, Ref. 34 reports that some of the matrices are unsymmetric and thus this paper is excluded from consideration herein. Until very recently the only solution procedures used were of the second class, $\{\dot{f}\}=0$. It will be shown in the present study that this is one of the most inefficient procedures for the combined problem.

For purposes of discussion the equations of equilibrium for combined material-geometric nonlinearities may be written in the form

$$[K]\{q\}=\{P'\}+\{Q^P\}+\{Q^{NL}\} \tag{13}$$

where, $[K]$, $\{q\}$ and $\{P'\}$ are defined after Eq. (1), $\{Q^P\}$ = pseudoforces due to nonlinear material properties (not the same as $\{Q\}$ in Eq. (1) since expressions for the total strain include both geometric nonlinearities and plastic effects), and $\{Q^{NL}\}$ = pseudoforce due to geometric nonlinearities. The first derivative of Eq. (13) with respect to the load parameter is written in two separate forms

$$([K]-[\partial Q_i^{NL}/\partial q_j]-[\partial Q_i^{P}/\partial q_j])\{\dot{q}\}=\{\bar{P}\} \tag{14}$$

and

$$([K]-[\partial Q_i^{NL}/\partial q_j])\{\dot{q}\}=\{\dot{Q}^P\}+\{\bar{P}\} \tag{15}$$

The matrices $[\partial Q_i^{NL}/\partial q_j]$ and $[\partial Q_i^{P}/\partial q_j]$ are contributions to the tangent stiffness matrix and are denoted as

$$[K^{NL}]=-[\partial Q_i^{NL}/\partial q_j] \tag{16}$$

$$[K^{P}]=-[\partial Q_i^{P}/\partial q_j] \tag{17}$$

In Refs. 28, 32, 36, and 37 the solution of Eq. (14) is obtained using a simple forward difference expression. Hofmeister et al.[32] control the amount of drift by applying a Newton-Raphson iteration ($f=0$) after a specified number of increments. Felippa[28] formulates the problem in the deformed coordinates of the body. In Ref. 36 the deformed coordinates are referred to but the effects of the deflections and rotations are neglected in the transformation matrix from local to global coordinates. The rigorous and correct derivation of the problem in the Lagrangian coordinates is given by Haisler.[5] In effect the deformed coordinates should not have been referred to in Ref. 36, but this oversight has no bearing on the accuracy of the results.

Armen, Pifko, and Levine[35] solved the combined problem through Eq. (15) where a simple backwards difference expression is used for $\{\dot{Q}^P\}$. The rate of convergence with load increment is much slower for Eq. (15) than for Eq. (14).

Another possible form of $\{\dot{f}\}=0$, other than that given in Eqs. (14) and (15), is to include the $\{\dot{Q}^{NL}\}$ term on the right-hand side as pseudoforces in the same manner as $\{\dot{Q}^P\}$ is written in Eq. (15). This form is very appealing as it requires only a single inversion of the stiffness matrix. However, a very exhaustive search for a stable numerical procedure presented in Ref. 5 has shown this form to be numerically unstable for significant nonlinearities.

Formulation

The purpose of this section is to summarize the formulation for the combined large deflection, elastic-plastic problem. The present research is based on equilibrium equations as presented by Haisler,[5] which are written in undeformed coordinates and valid for large deflections and large strains (although small strains are assumed herein). In terms of pseudoforces, the equations of equilibrium take the form given previously by Eq. (13). The formulation is valid for any incremental law of plasticity although the current results make use of the Von Mises yield criteria, isotropic hardening, and elastic unloading. A piecewise linear strain-hardening material curve is assumed. The formulation of the plasticity theory parallels that given by Marcal.[36] In order to follow the stress-strain curve as closely as possible, provision is made for dividing each increment of strain into subincrements. The treatment of the transition region between elastic and plastic behavior is the same one presented by Krieg and Duffey.[38] Since it is the primary purpose of this paper to evaluate the solution procedures, the details of the above formulation are omitted here but may be found in Ref. 1. In particular, Ref. 1 presents explicit equations for the pseudoforce and tangent stiffness terms for arbitrary shells with special attention being given to shells of revolution.

Solution Procedures and Their Evaluation

This section presents a comparative study of the solution techniques applicable to the equilibrium equations given by Eq. (13). The primary objective is to point out the advantages and shortcomings of each solution procedure. Although application is made to plates and shells of revolution, it is felt that the conclusions reached regarding each procedure would, in general, be valid for a wide class of problems.

Checkout Problem

Before going into a detailed comparison of the solution procedures, it was necessary to check the entire formulation and computer code by solving some typical problems and comparing the results with known solutions.

The large elastic-plastic deflection solution of a torispherical shell, which has been analyzed previously by Yaghmai,[37] is shown in Fig. 1. The material of the shell is elastic-perfectly plastic with a yield stress $\sigma_y=30{,}000$ psi, Young's modulus $E=30\times10^6$ psi, and Poisson's ratio $\nu=0.3$.

Figure 1 presents the results from Ref. 37 for two different load increments using the tangent stiffness solution procedure (no correction term) and three sets of results obtained herein by the Newton-Raphson, incremental tangent stiffness ($\dot{f}=0$), and first-order self-correcting procedure ($\dot{f}+Zf=0$). These procedures will be written out in more explicit form in the next section.

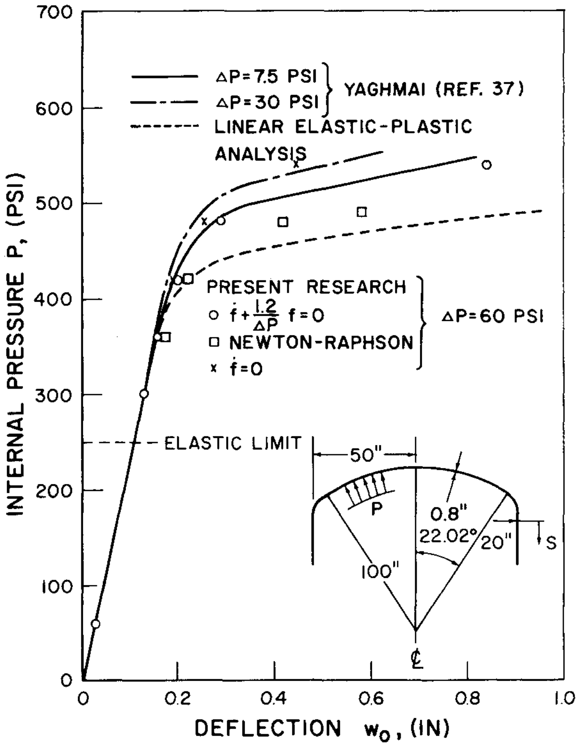

Fig. 1 The normal deflection at the apex, w_o.

The results from the Newton-Raphson procedure must be considered the correct solution for a load increment of 60 psi. However, it should be emphasized that the converged solution depends on the increment of load being used. From Fig. 1 it is rather obvious that the first-order self-correcting procedure gives considerably better results than the purely incremental stiffness approach. Further the results are in reasonable agreement with the results reported by Yaghmai.[37]

Test Problem

For the test case, a mild steel plate shown in Fig. 2 was chosen. A uniaxial yield stress of 36,000 psi was used with perfectly plastic behavior up to an equivalent plastic strain of 0.011 in/in. Next a secondary modulus of 70,000 psi was used to an equivalent plastic strain of 0.05 in./in. Beyond an equivalent plastic strain of 0.05, the material was assumed to be perfectly plastic. However, the yield stress of the first two elements adjacent to the load was raised to 100,000 and 50,000 psi, respectively, and the uniaxial stress-strain curve translated upward accordingly.

The load deflection curves for the test case as obtained by the Newton-Raphson procedure and by the first order self-correcting procedure are shown in Fig. 2. The two numbers within the

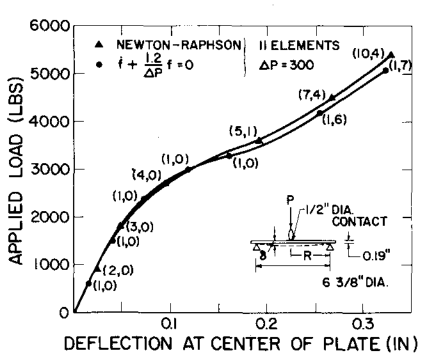

Fig. 2 Load-deflection curve for flat plate.

parentheses indicate the number of iterations required and the number of stations unloading, respectively. It is noted that a small degree of unloading (7 stations) occurs in this problem. It was hoped that a more pronounced unloading would occur to serve as a critical test for the solution procedures.

More detailed studies comparing stresses and deflections are presented in Ref. 2. Based on these results, it was decided to use the accuracy of the load deflection curve as a means of evaluating the different solution procedures. The curve referred to as the converged solution is the solution which converged with load increment refinement and was obtained by two different solution procedures.

Exact Solution, $\{f\} = \{0\}$

The first class of solution procedures is that in which the equations of equilibrium are satisfied exactly. Here the function $\{f\}$ is defined as

$$\{f\} = -[K]\{q\}+P\{\bar{P}\}+\{Q^{NL}\}+\{Q^{P}\} \tag{18}$$

where $\{Q^{NL}\}$ = pseudoforces due to geometric nonlinearities and $\{Q^P\}$ = pseudoforces due to material nonlinearities. The first procedure tested was the Newton-Raphson procedure. For this procedure the load is increased in increments and a value for the first guess of the displacement is obtained by some extrapolation procedure. For these values of the displacements the function $\{f(q_0)\}$ is computed. After this a value of $\{\Delta q\}$ is sought such that $\{f(q_0+\Delta q)\} = \{0\}$. Expanding this latter expression into a first-order Taylor's series yields the expression for $\{\Delta q\}$

$$([K]+[K^{NL}]+[K^P])\{\Delta q\} = \{f(q_0)\} \tag{19}$$

where $[K^{NL}]$ and $[K^P]$ are defined by Eqs. (16) and (17). $\{\Delta q\}$ is determined by solving Eq. (19) and the next guess for $\{q\}$ is $\{q_0+\Delta q\}$. The force unbalance $\{f\}$ and matrices $[K^{NL}]$ and $[K^P]$ are updated and a new value of $\{\Delta q\}$ is determined by solving Eq. (19) again. The solution process is continued until hopefully the process converges.

The results obtained for the test problem by the Newton-Raphson procedure are presented in Fig. 2. However, it is noted that the maximum load obtained is 5400 lb. At this load the procedure failed to converge in 10 iterations. The load increment was automatically reduced twice by a factor of 4 and each time failed to converge in ten additional iterations.

The Newton-Raphson procedure was the first procedure tested and consequently a considerable effort was expended to overcome the convergence difficulties. The following were tried.

1) Under-relaxation—An under-relaxation factor was applied to the displacement increments at each cycle of iteration. The relaxation factor was reduced by 0.2 (from a value of 1.0) every three cycles with the minimum value being restricted to 0.5. This process yielded one additional increment of load after the load increment had been reduced by a factor of 4. Thus for practical purposes the process did not yield any significant improvement in convergence characteristics.

2) Second-order extrapolation—For most cases the first guess was obtained using a linear extrapolation of the displacement. A quadratic extrapolation procedure was used and it was found that the process converged in fewer iterations. However, it did not converge for a load higher than 5400 lb.

3) Reduction of load increment—The load increment was reduced to 150 lb and a second order extrapolation procedure was used for the first trial values. This yielded converged results for a load of 5750 lb. However, this procedure failed to converge for higher loads.

4) Modified Newton-Raphson—The procedure was modified so that the nonlinear stiffness matrix and plasticity matrix were updated at the beginning of the increment and only every 4 cycles thereafter. This process shortened the computer time required but again the process failed to converge for a load above 5400 lb.

Close examination of the iterational process reveals that the trial values oscillate between values which cause loading and

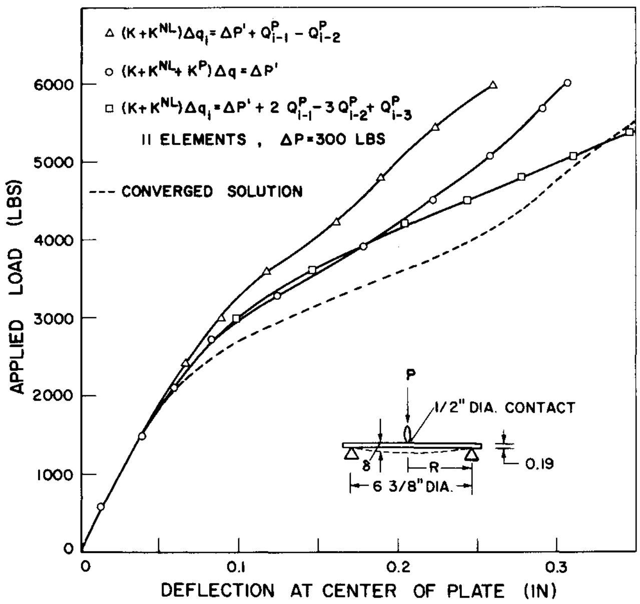

Fig. 3 Load-deflection curves for $\dot{f} = 0$.

unloading respectively. The process does not diverge but simply oscillates.

It should be noted that the matrix $[K^P]$ can have discrete discontinuities when unloading occurs. Since all the convergence proofs for the Newton-Raphson procedure assume a continuous first derivative, it is not possible to state specifically that the procedure will converge when there is a possibility of loading with elastic unloading. Furthermore under such conditions there is no unique solution for the deflections. All these considerations lead to the conclusion that additional refinements are needed in the Newton-Raphson solution procedure to make it applicable to problems involving complex combined loading-unloading behavior.

The very simplest form of the modified Newton-Raphson procedure is obtained when the matrices $[K^{NL}]$ and $[K^P]$ are set equal to zero. For this case the process reduces to the method of successive approximations as given by Eq. (3). This method was tried on the test problem to only a limited degree as it is a special form of the Newton-Raphson procedure. Results by successive approximations indicate that the convergence rate is very slow for large plastic strains and that convergence does not occur for elastic unloading. The convergence rate was so very slow in some cases that the difference between successive values of the deflections was so small as to indicate false convergence. The slow convergence rate has previously been pointed out in Ref. 13.

Incremental Stiffness, $\{\dot{f}\} = \{0\}$

As will be shown in this section a special case of $\{\dot{f}\} = \{0\}$ is the incremental stiffness method which has been so popular in nonlinear analyses. However, most researchers in nonlinear analyses have already realized that considerably better results are obtained when the first-order self-correcting form is used. Consequently, only a short discussion will be presented for $\{\dot{f}\} = \{0\}$.

Taking the derivative of Eq. (18) with respect to the scalar load parameter P yields

$$[K]\{\dot{q}\} = \{\dot{\bar{P}}\} + \{\dot{Q}^{NL}\} + \{\dot{Q}^P\} \tag{20}$$

where the dot indicates differentiation with respect to P.

The simplest solution procedure for Eq. (20) is an Euler forward difference for $\{\dot{q}\}$ and an Euler backwards difference for $\{\dot{Q}^{NL}\}$ and $\{\dot{Q}^P\}$. This form may be used for moderately nonlinear problems but a rather exhaustive study by Haisler[5] has shown that such a representation of $\{\dot{Q}^{NL}\}$ is very unstable. For this reason no studies were conducted for $\{\dot{f}\} = \{0\}$ with all pseudoforces on the right-hand side.

Applying chain rule differentiation to $\{\dot{Q}^{NL}\}$ and to $\{\dot{Q}^P\}$ two other forms of Eq. (20) are obtained.

$$([K]+[K^{NL}]+[K^P])\{\dot{q}\} = \{\dot{\bar{P}}\} \tag{21}$$

$$([K]+[K^{NL}])\{\dot{q}\} = \{\dot{\bar{P}}\} + \{\dot{Q}^P\} \tag{22}$$

Equation (21) is a first-order nonlinear differential equation which may be solved by any one of a large number of solution procedures such as predictor-corrector and Runge-Kutta methods. The only method considered herein is the simple Euler forward difference which reduces to the incremental stiffness approach. Results for this method applied to the test problem with a load increment of 300 lb are shown in Fig. 3. It is noted that the theoretical solution tends to drift appreciably from the converged solution.

Also shown in Fig. 3 are the results for the solution of Eq. (22) using an Euler forward difference for $\{\dot{q}\}$ and a backwards difference for $\{\dot{Q}^P\}$. This solution procedure is quite inaccurate for a load increment of 300 lb.

The remaining result in Fig. 3 is for the solution of Eq. (22) with $\{\dot{Q}^P\}$ being determined by a linear extrapolation procedure. This procedure improves the accuracy of the load deflection curve; but close examination of the data reveals that an unrealistic elastic unloading is obtained. The same type of extrapolation procedure was tried with the first order self-correcting form but is not presented due to erroneous results for unloading.

First-Order Self-Correcting, $\{\dot{f}\} + Z\{f\} = \{0\}$

This particular solution procedure is referred to as a first order self-correcting solution procedure. Applying this formula to the definition of $\{f\}$ as given by Eq. (18) and using chain rule differentiation of both $\{\dot{Q}^{NL}\}$ and $\{\dot{Q}^P\}$ yields

$$([K]+[K^{NL}]+[K^P])\{\dot{q}\} = \{\dot{\bar{P}}\} + Z\{f\} \tag{23}$$

While Eq. (23) may be solved by any numerical integration method, the only procedure considered here is the Euler forward difference expression for $\{\dot{q}\}$. For this procedure Eq. (23) reduces to

$$([K]+[K^{NL}]+[K^P])\{\Delta q\} = \{\Delta P'\} + Z\Delta P\{f\} \tag{24}$$

Comparing Eq. (24) with Eq. (19) for the Newton-Raphson procedure reveals a great degree of similarity. In particular for $Z\Delta P = 1.0$ the right-hand side represents the unbalance in forces due to $\{\Delta P'\}$ and the total unbalance at the beginning of the load increment. This has led to naming the procedure an incremental procedure with a one step Newton-Raphson correction.[33] Experiences for $Z\Delta P = 1$ have shown a considerable improvement over the purely incremental approach. However for large load increments the procedure still tends to drift.

The above explanation may be used to develop an intuitive feel for the proper selection of $Z\Delta P$. In the first place since $\{f\}$ is the value of the force unbalance at the beginning of the increment, then $\{f\}$ at the end of the increment will be slightly larger and thus a factor of $Z\Delta P > 1$ is indicated. However if the Newton-Raphson approach is converging by oscillating about the true solution it is clear that a large value of $Z\Delta P$ would

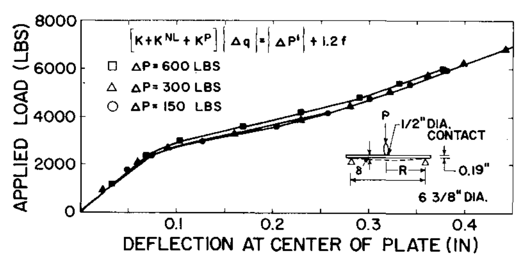

Fig. 4 Convergence studies with load refinement.

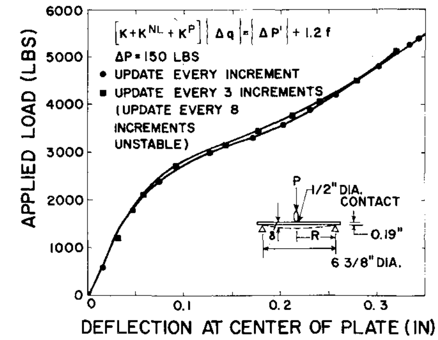

Fig. 5 Effect of updating stiffness matrix on load-deflection curve.

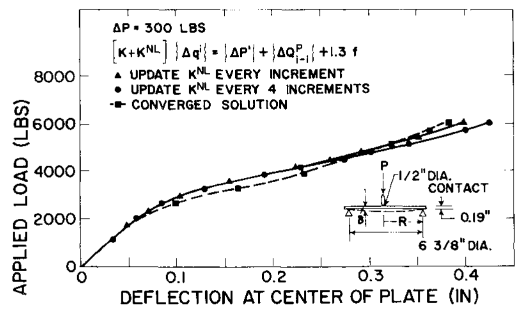

Fig. 7 Effects of updating K^{NL} on accuracy of load-deflection curve.

cause the solution procedure to diverge. Based on this argument and experience a value of $Z\Delta P$ of 1.2–1.3 is believed to be conservative from a stability point of view and to increase the accuracy somewhat. Levine[39] recently conducted a numerical evaluation of using $Z\Delta P > 1$. Levine reports no appreciable difference between results for $Z\Delta P = 1$ and $Z\Delta P = 1.3$. Similar recent studies by the present authors seem to confirm the conclusions reached by Levine. Thus, in summary there are some doubts about the advantages of using $Z\Delta P > 1$ but little doubt about the increased accuracy obtained in going from $Z\Delta P = 0.0$ (incremental stiffness) to $Z\Delta P = 1.0$.

Results for the load-deflection curve of the test problem with $Z\Delta P = 1.2$ are presented in Fig. 4. The results show that good accuracy is obtained using only ten load increments and for all practical purposes the results have converged for 20 load increments.

As the formulation given by Eq. (24) is self-correcting it would appear that it is not necessary to update $[K^{NL}]$ and $[K^P]$ every load increment. This was studied with the results being presented in Fig. 5. It is noted from Fig. 5 that 40 total increments with 13 updates in the coefficient matrix does not change the accuracy appreciably. However, comparing the results in Fig. 4 for 10 increments with the results in Fig. 5 with 13 updates reveals comparable degrees of accuracy. These results indicate that it does not make much difference in accuracy whether one uses a small number of increments and updates every time or uses a large number and only updates so that the total number of updates is about the same.

Another form of $\dot{f} + Zf = 0$ is obtained by using pseudoforces for $\{\dot{Q}^P\}$ and the nonlinear stiffness matrix approach for $\{\dot{Q}^{NL}\}$. Again the pseudoforce approach was not attempted for $\{\dot{Q}^{NL}\}$ as previous studies have indicated numerical instability problems. Using a forward difference for $\{\dot{q}\}$ and a backwards difference for $\{\dot{Q}^P\}$ yields

$$([K]+[K^{NL}])\{\Delta q_i\} = \{\Delta P^i\} + \{\Delta Q_{i-1}{}^P\} + Z\Delta P\{f\}_{i-1} \quad (25)$$

The test problem results using $Z\Delta P = 1.3$ are shown in Fig. 6 for three different load increments. The more refined results agree with those obtained through Eq. (24). Figure 6 reveals that reasonably accurate results are obtained when 60 increments of load are used. Figure 7 shows how the results are influenced by not updating $[K^{NL}]$ every load increment. It is noted that only the values at large values of the load are changed. This occurs simply because geometric nonlinearities are not significant for small deflections.

A comparison of the accuracy obtained through Eqs. (24) and (25) is shown in Fig. 8 for 20 load increments. It is observed that the results obtained through Eq. (25) are not nearly as good as the results obtained using Eq. (24). For this reason the solution for combined geometric and material nonlinearities would appear to best be achieved through Eq. (24) since the coefficient matrix must be reformed and inverted in both procedures. This of course depends on the formulation since in some formulations it is very tedious to compute $[K^P]$. For nonlinearities due only to plastic deformations the solution through Eq. (25) appears to be very promising as it involves only a single inversion of the matrix $[K]$.

Second-Order Self-Correcting, $\{\ddot{f}\} + C\{\dot{f}\} + Z\{f\} = \{0\}$

This particular form, which is called the second order self-correcting form, is a result of a long and tedious search for a method of solution for geometric nonlinearities which requires only a single inversion of the stiffness matrix and is very economical on computer storage requirements. It is a natural extension of the first order self-correcting form and is very appealing since the equation is the same as for damped harmonic motion.

Two different forms of this equation have been explored. In the first (Ref. 31) the expression for f [Eq. (18)] is substituted into the equation $\ddot{f} + C\dot{f} + Zf = 0$ and certain terms collected to yield the equation for damped harmonic motion in terms of the displacement q. This form is then solved by an implicit four point backwards difference formula for the displacements as a function of the load P. The resulting solution oscillates about the true solution for geometric nonlinearities which is undesirable for material nonlinearities.

The second form[30] of the equation is obtained as follows and was studied herein. The particular form is for $C = 0$. First Eq. (18) is rearranged in the form

$$[K]\{q_i\} = P_i\{\bar{P}\} + \{Q_{i-1}{}^P\} + \{Q_{i-1}{}^{NL}\} - \{f_i\} \quad (26)$$

It is noted that the pseudoforce terms are used as their known

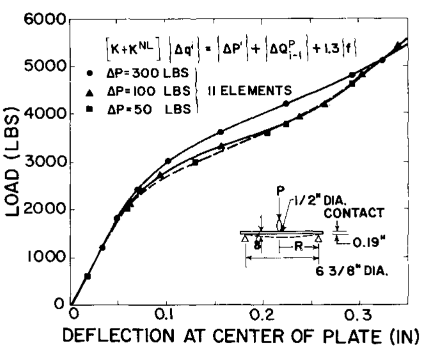

Fig. 6 Load-deflection curves for $(K + K^{NL})\Delta q^i = \Delta P^1 + \Delta Q_{i-1}{}^P + 1.3f$.

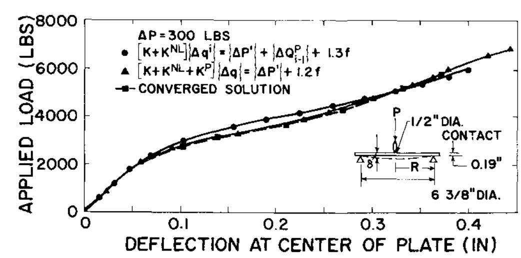

Fig. 8 Effects of representation of plasticity term on the accuracy of load-deflection curve.

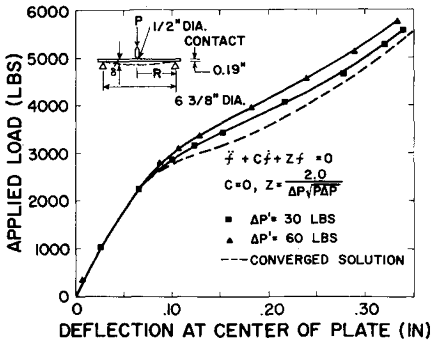

Fig. 9 Convergence of load-deflection curve for $\ddot{f}+C\dot{f}+Zf=0$.

values at $i-1$. The unbalance in force f_i is determined from the equation

$$\{\ddot{f}\}+Z\{f\}=0 \tag{27}$$

which is

$$\{f\}=\{A\}\cos(Z)^{1/2}s+\{B\}\sin(Z)^{1/2}s \tag{28}$$

where s has the range from 0.0 to ΔP. The matrices of coefficients $\{A\}$ and $\{B\}$ are determined from the known values of $\{f\}$ and $\{\dot{f}\}$ at the beginning of the increment

$$\{f(s=0)\}=\{A\}=-[K]\{q_{i-1}\} + P_{i-1}\{\bar{P}\}+\{Q_{i-1}{}^{P}\}+\{Q_{i-1}{}^{NL}\} \tag{29}$$

$$\{\dot{f}(s=0)\}=(Z)^{1/2}\{B\}=-[K]\{\dot{q}_{i-1}\}+\{\bar{P}\} \tag{30}$$

The derivatives of $\{Q^P\}$ and $\{Q^{NL}\}$ are omitted in Eq. (30) as they are held constant over the increment as seen in Eq. (26).

Substituting the values for $\{A\}$ and $\{B\}$ given by Eq. (29) and (30) into Eq. (28) using a backwards difference for $\{\dot{q}\}$, and using the results in Eq. (26) yields the following recursion relation for $\{q_i\}$ in terms of known quantities

$$\{q_i\}=A_1\{q_{i-2}\}+A_2\{q_{i-1}\}+[K]^{-1}(A_3\{\bar{P}\}+ A_4\{Q_{i-1}{}^{NL}\}+A_5\{Q_{i-1}{}^{P}\}) \tag{31}$$

where

$$\begin{aligned} A_1 &= -\sin(\zeta)/\zeta \\ A_2 &= \cos(\zeta)+\sin(\zeta)/\zeta \\ A_3 &= P_i-\cos(\zeta)P_{i-1}-\sin(\zeta)/(Z)^{1/2} \\ A_4 &= 1-\cos(\zeta) \\ A_5 &= A_4 \end{aligned} \tag{32}$$

and

$$\zeta=(Z)^{1/2}\Delta P$$

It should be noted that the application of the recurrence relation given by Eq. (31) requires only the symmetric stiffness matrix and 6 one-dimensional arrays.

The maximum value of P is chosen to be 100 and Z is varied in accordance to

$$Z=2.0/\Delta P(P\Delta P)^{1/2} \tag{33}$$

Results for the test problem using the second-order self-correcting form are shown in Fig. 9. It is noted that 200 load increments are needed to give a reasonably accurate solution. However, only a single inversion of the symmetric stiffness matrix is required.

It should be pointed out that the stability of this procedure is controlled appreciably by the parameters C and Z in addition to the step size ΔP. Massett and Stricklin[30] have indicated that a small positive value of C enhances the stability but causes the response to be somewhat sluggish. The possibility exists of using a small negative value of C in conjunction with a small value of Z as given by Eq. (33). It would appear that the negative C would increase the response and hopefully the small value of Z would retain the stability. This approach is possible since the load increment ΔP traverses only a small portion of the complete response history. This is, of course, one of several possibilities that should be investigated.

Discussion of Solution Procedures

In this section the more promising solution procedures for geometric and/or material nonlinearities are summarized. However, before summarizing these procedures based on the studies presented herein, it should be emphasized that there is ample room for further studies. In particular the forms for both the first and second derivative formulations should be studied. Furthermore, the solution procedures presented herein should be subjected to a more severe test, e.g., when a large percentage of the structure transcends from a state of loading to elastic unloading.

Geometric Nonlinearities

For problems involving geometric nonlinearities there are three procedures worth discussing. The first is the modified Newton-Raphson procedure[6] which is the only method capable of yielding the exact solution. This procedure requires some updating of $[K^{NL}]$ and a very large number of iterative cycles where $\{Q^{NL}\}$ is evaluated each time.

The second method is the second-order self-correcting form as given in Refs. 30 and 31. This procedure is very efficient on storage requirements and involves only a single inversion of the stiffness matrix. However, a large number of load increments must be used for accurate results which implies many calculations of the pseudoforce terms $\{Q^{NL}\}$.

The third method is the first-order self-correcting form.[6] This procedure has previously been evaluated for $Z\Delta P=1.0$ and was found to require many increments of load to achieve a converged solution. As each increment requires the forming of $[K^{NL}]$ and the solution of a system of equations, the method requires considerable computer time.

Material Nonlinearities

For material nonlinearities only, the consensus of opinion of the writers is that the first-order self-corretcing procedure is quite outstanding. Both forms as given by Eqs. (24) and (25) yield good results. Equation (25) is particularly appealing as a simple recursion relation may be developed. This recursion relation is given by

$$\{q_{i+1}\}=B_1\{q_i\}+[K]^{-1}(B_2\{\bar{P}\}+B_3\{Q_i{}^{P}\}+B_4\{Q_{i-1}{}^{P}\}) \tag{34}$$

where

$$B_1=1-Z\Delta P,\quad B_2=\Delta P(1+ZP_i),\quad B_3=1+Z\Delta P,\quad B_4=-1 \tag{35}$$

The recursion relation given by Eq. (34) conserves computer storage space, requires one inversion of the matrix $[K]$, and yields good results for a relatively small number of load increments as can be seen in Fig. 6. The solution procedure as given by Eq. (24) requires fewer load increments but requires the computation of $[K^P]$ and the solution of a set of algebraic equations for each load increment.

Combined Geometric and Material Nonlinearities

Two solution procedures are suited for the analysis of combined geometric and material nonlinearities. The first is the first-order self-correcting procedure in the form given by Eq. (24). The other procedure is the second-order self-correcting procedure using the recurrence relation given by Eq. (31).

Comparing the results in Figs. 4 and 9 reveals that comparable accuracy is obtained using 10 increments in the first-order self-correcting or 200 increments in the second-order self-correcting procedures. Even when a relatively large number of increments is used, the second-order self-correcting procedure is quite efficient since it requires only a single inversion of the stiffness matrix. The required number of increments for comparable accuracy would be shifted considerably for problems involving predominantly geometric nonlinearities. The shift would be to the benefit of the second-order self-correcting procedure.

References

[1] Stricklin, J. A., Haisler, W. E., and Von Riesemann, W. A., "Computation and Solution Procedures for Nonlinear Analysis by Combined Finite Element—Finite Difference Methods," National Symposium on Computerized Structural Analysis and Design, George Washington Univ., Washington, D.C., March 27–29, 1972.

[2] Stricklin, J. A., Haisler, W. E., and Von Riesemann, W. A., "Formulation, Computation, and Solution Procedure for Material and/or Geometric Nonlinear Structural Analysis by the Finite Element Method," Rept. SC-CR-72-3102, Jan. 1972, Sandia Labs., Albuquerque, N. Mex.

[3] Martin, H. C., "Finite Element Formulation of Geometrically Nonlinear Problems," Paper US 2-2, *Proceedings of Japan-U.S. Seminar on Matrix Methods in Structural Analysis and Design*, Tokyo, 1969, pp. 1–53.

[4] Oden, J. T., "Finite Element Applications in Nonlinear Structural Analysis," *Proceedings of the Conference on Finite Element Method*, Vanderbilt Univ., Nashville, Tenn. Nov. 1969, pp. 419–457.

[5] Haisler, W. E., Jr., "Development and Evaluation of Solution Procedures for Nonlinear Structural Analysis," Ph.D. dissertation, Dec. 1970, Texas A & M Univ., College Station, Texas.

[6] Haisler, W. E., Stricklin, J. A., and Stebbins, F. J., "Development and Evaluation of Solution Procedures for Geometrically Nonlinear Structural Analysis," *AIAA Journal*, Vol. 10, No. 3, March 1972, pp. 264–272.

[7] Stricklin, J. A., Haisler, W. E., and Von Riesemann, W. A., "Geometrically Nonlinear Structural Analysis by the Direct Stiffness Method," Rept. 70-16, Aug. 1970, Aerospace Engineering Dept., Texas A & M Univ., College Station, Texas.

[8] Mendelson, A. and Manson, S. S., "Practical Solution of Plastic Deformation Problems in the Elastic-Plastic Range," TR R28, 1959, NASA.

[9] Argyris, J. H., "Elasto-Plastic Matrix Displacement Analysis of Three-Dimensional Continua," *Journal of the Royal Aeronautical Society*, Vol. 69, No. 657, Sept. 1965, pp. 633–636.

[10] De Donato, O., "Iterative Solution of the Incremental Problem for Elastic-Plastic Structures with Associated Flow Laws," *International Journal of Solids and Structures*, Vol. 5, 1969, pp. 81–95.

[11] Havner, K. S., "On the Formulation and Iterative Solution of Small Strain Plasticity Problems," *Quarterly of Applied Mathematics*, Vol. 23, No. 4, Jan. 1966, pp. 323–335.

[12] Marcal, P. V., "A Comparative Study of Numerical Methods of Elastic-Plastic Analysis," *AIAA Journal*, Vol. 6, No. 1, Jan. 1968, p. 157.

[13] Witmer, E. A. and Kotanchik, J. J., "Progress Report on Discrete-Element Elastic and Elastic-Plastic Analysis of Shells of Revolution Subjected to Axisymmetric and Asymmetric Loading," *Proceedings of the Second Conference on Matrix Methods in Structural Mechanics*, U.S. Air Force, Oct. 15–17, 1968, pp. 1341–1453; also AFFDL-TR-68-150, Dec. 1969, Wright-Patterson Air Force Base, Ohio.

[14] Zienkiewicz, O. C., Valliappan, S., and King, I. P., "Elasto-Plastic Solutions of Engineering Problems: 'Initial Stress' Finite Element Approach," *International Journal for Numerical Methods in Engineering*, Vol. 1, No. 1, Jan.–March. 1969, pp. 75–100.

[15] Zienkiewicz, O. C. and Nayak, G. C., "Elastic-Plastic Stress Analysis with Curved Isoparametric Elements for Various Constitutive Relations," (Paper in preparation) Aug. 1971.

[16] Oden, J. T. and Kubitza, W. K., "Numerical Analysis of Nonlinear Pneumatic Structures," *Proceedings of the First International Colloquium on Pneumatic Structures*, Stuttgart, May 1967, pp. 87–107.

[17] Stanton, E. L. and Schmit, L. A., "A Discrete Element Stress and Displacement Analysis of Elasto-Plastic Plates," *AIAA Journal*, Vol. 8, No. 7, July 1970, pp. 1245–1251.

[18] Gallagher, R. H., Padlog, J., and Bijlaard, P. P., "Stress Analysis of Heated Complex Shapes," *Journal of the American Rocket Society*, Vol. 32, No. 5, May 1962, pp. 700–707.

[19] Armen, H., Jr., Isakson, G., and Pifko, A., "Discrete Element Methods for the Plastic Analysis of Structures Subjected to Cyclic Loading," *AIAA/ASME 8th Structures, Structural Dynamics and Materials Conference*, Palm Springs, Calif., March 1967, pp. 148–161.

[20] Isakson, G., "Discrete-Element Plastic Analysis of Structures in a State of Modified Plane Strain," *AIAA Journal*, Vol. 7, No. 3, March 1969, pp. 545–547.

[21] Isakson, G., Armen, H., Jr., and Pifko, A., "Discrete-Element Methods for Plastic Analysis of Structures," CR-803, Oct. 1967, NASA.

[22] Pope, G. G., "The Application of the Matrix Displacement Method in Plane Elasto-Plastic Problems," *Proceedings of the Conference on Matrix Methods in Structural Mechanics*, U.S. Air Force, Oct. 26–28, 1965, pp. 635–654; also AFFDL-TR-66-80, Nov. 1966, Wright-Patterson Air Force Base, Ohio.

[23] Swedlow, J. L. and Yang, W. H., "Stiffness Analysis of Elastic-Plastic Plates," Ph.D. dissertation, 1965, California Inst. of Technology.

[24] Marcal, P. V. and King, J. P., "Elastic-Plastic Analysis of Two-Dimensional Stress Systems by the Finite Element Method," *International Journal of the Mechanical Sciences*, Vol. 9, 1967, pp. 142–155.

[25] Yamada, Y., Kawai, T., Yoshimura, N., and Sakurai, T., "Analysis of the Elastic-Plastic Problem by the Matrix Displacement Method," *Proceedings of the Second Conference on Matrix Methods in Structural Mechanics*, U.S. Air Force, Oct. 15–17, 1968, pp. 1271–1299; also AFFDL-TR-68-150, Dec. 1969, Wright-Patterson Air Force Base, Ohio.

[26] Khojasteh-Bakht, M., "Analysis of Elastic-Plastic Shells of Revolution under Axisymmetric Loading by the Finite Element Method," SESM 67-8, Ph.D. dissertation, April 1967, Structural Engineering Lab., Univ. of California, Berkeley, Calif.

[27] Khojasteh-Bakht, M. and Popov, E. P., "Analysis of Elastic-Plastic Shells of Revolution," *Journal of the Engineering Mechanics Division*, ASCE, Vol. 96, No. EM3, June 1970, pp. 327–340.

[28] Felippa, C. A., "Refined Finite Element Analysis of Linear and Nonlinear Two-Dimensional Structures," SESM 66-22, Oct. 1966, Structural Engineering Lab., Univ. of California, Berkeley, Calif.

[29] Richard, R. M. and Blacklock, J. R., "Finite Element Analysis of Inelastic-Structures," *AIAA Journal*, Vol. 7, No. 3, March 1969, pp. 432–438.

[30] Massett, D. A. and Stricklin, J. A., "Self-Correcting Incremental Approach in Nonlinear Structural Mechanics," *AIAA Journal*, Vol. 9, No. 12, Dec. 1971, pp. 2464–2466.

[31] Stricklin, J. A., Haisler, W. E., and Von Riesemann, W. A., "Self-Correcting Initial Value Formulations in Nonlinear Structural Mechanics," *AIAA Journal*, Vol. 9, No. 10, Oct. 1971, pp. 2066–2067.

[32] Hofmeister, L. D., Greenbaum, G. A., and Evensen, D. A., "Large Strain Elasto-Plastic Finite Element Analysis," *AIAA Journal*, Vol. 9, No. 7, July 1971, pp. 1248–1254.

[33] Stricklin, J. A., Haisler, W. E., and Von Riesemann, W. A., "Geometrically Nonlinear Structural Analysis by the Direct Stiffness Method," *Transactions of the ASCE, Journal of the Structural Division*, Vol. 97, No. ST9, Sept. 1971, pp. 2299–2314.

[34] Akyuz, F. A. and Merwin, J. E., "Solution of Nonlinear Problems of Elastoplasticity by the Finite Element Method," *AIAA Journal*, Vol. 6, No. 10, Oct. 1968, pp. 1825–1831.

[35] Armen, H., Jr., Pifko, A., and Levine, H. S., "A Finite Element Method for the Plastic Bending Analysis of Structures," AFFDL-TR-68-150, *Proceedings of the Second Conference on Matrix Methods in Structural Mechanics*, Wright-Patterson Air Force Base, Ohio, Oct. 15–17, 1968.

[36] Marcal, P. V., "Large Deflection Analysis of Elastic-Plastic Shells of Revolution," *AIAA Journal*, Vol. 8, No. 9, Sept. 1970, pp. 1627–1633.

[37] Yaghmai, S., "Incremental Analysis of Large Deformation in Mechanics of Solids with Application to Axisymmetric Shells of Revolution," SESM 68-17, Dec. 1968, Structural Engineering Lab., Univ. of California, Berkeley, Calif.

[38] Krieg, R. D. and Duffey, T. A., "Univalve II: A Code to Calculate the Large Deflection Dynamic Response of Beams, Rings, Plates and Cylinders," Rept. SC-RR-68-303, Oct. 1968, Sandia Labs., Albuquerque, N. Mex.

[39] Levine, H. S., personal communication, 1972, Grumman Aerospace Corp., Bethpage, L.I.